BVP's: Shooting Methods for Linear Functions

Consider the Linear two-point BVP:

$$\begin{cases} x'' = \underbrace{u(t) + v(t)x + w(t)x'}^{f(t,x,x')} \\ x(a) = \alpha \quad , \quad x(b) = \beta \end{cases} \quad \text{——} (*)$$

Suppose that $u(t), v(t), w(t)$ are all cont. on $[a,b]$.

Now suppose we solve (as we do in shooting methods) the BVP $(*)$ by modifying it to an IVP to obtain the 2 sol'ns: $x_1(t)$ & $x_2(t)$

where $\begin{cases} x_1(a) = \alpha \\ x_1'(a) = z_1 \end{cases}$ $\begin{cases} x_2(a) = \alpha \\ x_2'(a) = z_2 \end{cases}$

Let $y(t) = \lambda x_1(t) + (1-\lambda)x_2(t)$

y solves $\begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha \ , \ y'(a) = \lambda z_1 + (1-\lambda) z_2 \end{cases}$

Now, we simply pick λ so that we solve the BVP, i.e., $y(b)=\beta$

$$y(b) = \lambda x_1(b) + (1-\lambda) x_2(b) = \beta$$

$$\Rightarrow \lambda = \frac{\beta - x_2(b)}{x_1(b) - x_2(b)}$$

So, to solve linear BVP's like (*)

(1) Solve the 2 IVP's (numerically)

$$\begin{cases} x'' = f(t,x,x') \\ x(a)=\alpha, \quad x'(a)=0 \end{cases} \Longrightarrow x_1(t_i) \quad i=1,2,\ldots$$

$$\begin{cases} x'' = f(t,x,x') \\ x(a)=\alpha, \quad x'(a)=1 \end{cases} \Longrightarrow x_2(t_i) \quad i=1,2,\ldots$$

(2) Set $\lambda = \dfrac{\beta - x_2(b)}{x_1(b) - x_2(b)}$

(3) The solution is approx. by
$y(t_i) = \lambda x_1(t_i) + (1-\lambda)\, x_2(t_i)$.

BVP's : Shooting/Newton's Method

Want to solve

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \quad x(b) = \beta \end{cases}$$

Instead we solve

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \quad x'(a) = z \end{cases} \qquad (1)$$

with solution $x_z(t)$ and

error $\boxed{\phi(z) = x_z(b) - \beta}$

↳ non-linear equation in z

$\Rightarrow$ Newton's method!

$$z_{n+1} = z_n - \frac{\phi(z_n)}{\phi'(z_n)}$$

Question

But we don't know $\phi(z)$ explicitly, how do we get $\phi'(z)$?

Answer

$$\phi(z) = x_z(b) - \beta \implies \boxed{\phi'(z) = \frac{\partial x_z(b)}{\partial z}}$$

where

$$\begin{cases} x_z'' = f(t, x, x') \\ x_z(a) = \alpha, \quad x_z'(a) = z \end{cases} \quad (2)$$

Great! But how do we get $\dfrac{\partial x_z(b)}{\partial z}$?

Answer

Differentiate (2) w.r.t. z!

$$\begin{cases} \dfrac{\partial x_z''}{\partial z} = \dfrac{\partial f}{\partial t} \cdot \cancelto{0}{\dfrac{\partial t}{\partial z}} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial z} + \dfrac{\partial f}{\partial x'} \cdot \dfrac{\partial x'}{\partial z} \\ \dfrac{\partial}{\partial z} x_z(a) = 0, \quad \dfrac{\partial}{\partial z} x_z'(a) = 1 \end{cases}$$

Rewriting this with $v := \frac{\partial x_z}{\partial z}$

$$\Rightarrow \begin{cases} v'' = \underbrace{\frac{\partial f^{(t, x_z, x_z')}}{\partial x_z}}_{\text{Can compute}} \cdot v + \frac{\partial f^{(t, x_z, x_z')}}{\partial x_z'} \cdot v' \\ v(a) = 0, \ v'(a) = 1 \end{cases} \quad \text{--- (3)}$$

This is an IVP, called the <u>first variational eq'n</u>

$$\boxed{\begin{array}{l} \text{So now, you (numerically) solve} \\ \text{(2) with initial cond } x'(a) = z_n \\ \text{and use this sol'n } (x_{z_n}) \text{ to} \\ \text{solve (3)} \Rightarrow \text{ have } \phi'(z_n) = v(b) \\ \qquad \Rightarrow \text{use in Newton's meth.} \\ \text{to get } z_{n+1} \text{ and repeat} \end{array}}$$

Multiple Shooting

Want to solve

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \quad x(b) = \beta \end{cases}$$

$\Rightarrow$ solve two IVP's

$$\begin{cases} x_1'' = f(t, x_1, x_1') \\ x_1(a) = \alpha, \; x_1'(a) = z_1 \end{cases} \qquad a \le t \le c$$

& $$\begin{cases} x_2'' = f(t, x_2, x_2') \\ x_2(b) = \beta, \quad x_2'(b) = z_2 \end{cases} \qquad c \le t \le b$$

$\hookrightarrow$ for this one, we decrease t

Idea: Adjust z_1, z_2 till the function $x(t) = \begin{cases} x_1(t) & t \in [a, c] \\ x_2(t) & t \in [c, b] \end{cases}$

solves the problem with

$$x_1(c) = x_2(c) \quad \& \quad x_1'(c) = x_2'(c)$$

We can thus define the function

$$\phi(z_1, z_2) = \begin{pmatrix} x_1(c) - x_2(c) \\ x_1'(c) - x_2'(c) \end{pmatrix}$$

both are functions of z_1, z_2

Want the non-linear f$\frac{t}{s}$

$$\phi(z_1, z_2) = 0$$

$\Rightarrow$ Newton's method (in 2 variables)

(170B)

Boundary Value Problems: Finite Differences

Idea: • Discretize the t-axis

$\rightarrow t_i \ , i = 1, 2, \ldots, n$

• Use approximations to the derivatives

Recall: $x'(t) \approx \dfrac{x(t+h) - x(t-h)}{2h}$

$O(h^2)$ error

$$x''(t) \approx \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}$$

$O(h^2)$ error

So now instead of solving

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha \ , \ x(b) = \beta \end{cases}$$

we solve (with $t_i = a + ih$, $i = 0, \ldots, n+1$

$y_i = y(t_i)$)

$$\begin{cases} y_0 = \alpha \\ \frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right), \\ \qquad \text{for } i = 1, ---, n \\ y_{n+1} = \beta \end{cases}$$

Due to the function f, this is potentially a non-linear system of equations in $y = (y_0 ---, y_{n+1})$

($\Rightarrow$ can use methods from 170B to solve it)

Finite Differences: The linear case

When

$$f(t, x, x') = u(t) + v(t)x + w(t)x'$$

we have

$$\begin{cases} y_0 = \alpha \\ \boxed{\frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) = u_i + v_i y_i + \frac{w_i}{h}(y_i - y_{i-1})} \\ y_{n+1} = \beta \end{cases}$$

$\Updownarrow$

(see next page)

linear in y

$$\underbrace{\left(-1-\tfrac{1}{2}hw_i\right)}_{a_{i-1}} y_{i-1} + \underbrace{\left(2+h^2v_i\right)}_{d_i} y_i + \underbrace{\left(-1+\tfrac{1}{2}hw_i\right)}_{c_i} y_{i+1} = \underbrace{-h^2u_i}_{b_i}$$

$\Rightarrow$ we can write this in matrix form

as $A\vec{y} = \vec{b}$ $\Leftrightarrow$

$$\underbrace{\begin{bmatrix} d_1 & c_1 & & & \\ a_2 & d_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & d_{n-1} & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 - a_0\overset{y_0}{\alpha} \\ b_2 \\ b_3 \\ \vdots \\ b_n - c_n\beta \end{pmatrix}}_{\text{Tri-diagonal} \Rightarrow \text{fast sol'n (170A)}}$$

"Theorem": If $v(t) > 0$ & $u, v, w \in C[a,b]$ then as $h \to 0$, sol'n of $Ay = b$ converges to sol'n of BVP

Theorem on Existence & Uniqueness of sol'ns to BVP's:

$$\begin{cases} x'' = f(t, x, x') \\ C_{11}\, x(a) + C_{12}\, x'(a) = C_{13} \\ C_{21}\, x(b) + C_{22}\, x'(b) = C_{23} \end{cases}$$

has a unique sol'n on $[a,b]$ if

(1) $f, \dfrac{\partial f}{\partial t}, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial x'}$ are conti on $D = [a,b] \times \mathbb{R}$

(2) $\dfrac{\partial f}{\partial x} > 0$, $\left|\dfrac{\partial f}{\partial x}\right| \leq M$, $\left|\dfrac{\partial f}{\partial x'}\right| \leq M$ on D

and

(3) $|C_{11}| + |C_{12}| > 0$

$|C_{21}| + |C_{22}| > 0$

$|C_{11}| + |C_{21}| > 0$

$C_{11} C_{12} \leq 0 \leq C_{21} C_{22}$

can use on most BVP's we've seen.

Topics covered in Midterm 2

* Everything till now.
* Focus on:

  * Multistep methods
    - Adams-Bashforth
    - Adams-Moulton
    - Explicit/implicit

  * Error Analysis
    - convergence/stability/consist.
    - Truncation error

  * Systems & Higher order ODE's
    - Taylor series methods
    - Other methods

  * BVP's:
    - Existence/Uniqueness
    - Shooting Methods
      * Linear eqs
      * Secant/Newton
    - Finite Differences